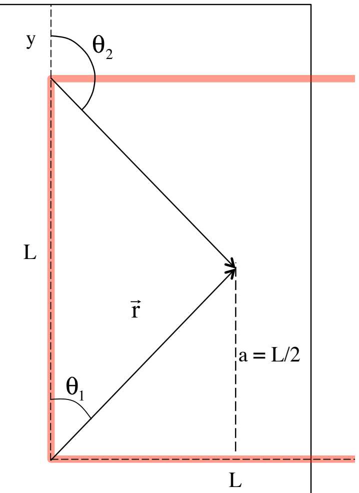
Problem 30.5

a.) The net magnetic field at the center generated by each side will be the same, so all we have to do is determine the field due to one side, then multiply by 4. If you tried to do this problem using Biot Savart as executed in Cartesian coordinates, the integral you end up with is a horror (I'm including it for your amusement at the end of this problem). Fortunately for you (and me), the book has been clever with trig functions and derived an expression (Equation 30.4) for the B-fld due to a short current carrying wire as:

$$B = \frac{\mu_o i}{4\pi a}\left(\cos\theta_1 - \cos\theta_2\right)$$

where the angles are defined as shown.



Executing that expression, we get:

$$\begin{aligned}
B &= 4\left[\frac{\mu_o i}{4\pi a}\left(\cos\theta_1 - \cos\theta_2\right)\right] \\
&= \frac{\mu_o i}{\pi(L/2)}\left(\cos\left(45^\circ\right) - \cos\left(135^\circ\right)\right) \\
&= \frac{2\mu_o i}{\pi L}\left(2(.707)\right) \\
&= \frac{2.83\mu_o i}{\pi L} \\
&= \frac{2.83\left(4\pi x10^{-7}\ T\bullet m/A\right)(10\ A)}{\pi(.4\ m)} \\
&= \left(28.3x10^{-6}\ T\right) \text{ into the page}
\end{aligned}$$

b.) For a circular coil:

$$4L = 4\pi R \quad \Rightarrow \quad 2R = {}^{4L}\!/_{\pi}$$

From the previous problem, we know the magnetic field function for a single turn coil:

$$\begin{aligned}
B &= \frac{\mu_o i}{2R} \\
&= \frac{\mu_o \pi i}{4L} \\
&= \frac{\left(4\pi x10^{-7}\ T\bullet m/A\right)(10\ A)}{\pi(.4\ m)} \\
&= 24.7x10^{-6}\ T
\end{aligned}$$

THIS ENDS THE PROBLEM.

IF YOU HAPPEN TO BE A MATH NERD, THOUGH, WITH A PARTICULARLY STRONG MASOCHISTIC BENT, READ ON.

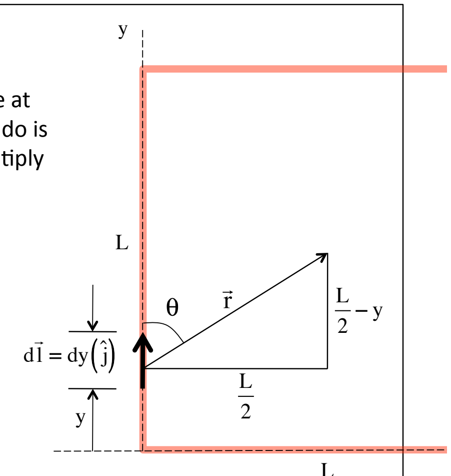
Problem 30.5 the hard way:

The net magnetic field generated by each side at the center will be the same, so all we have to do is determine the field due to one side, then multiply by 4. Biot Savart says:

$$dB = \frac{\mu_o i}{4\pi}\frac{d\vec{l}x\hat{r}}{r^2}$$

where the variables are defined in the sketch.

To begin with, we need to determine the magnitude of the vector "r" in terms of the variables given, and we need to determine the sine of the angle, also in those variables.



$$|\vec{r}| = \sqrt{\left(\frac{L}{2}\right)^2 + \left(\frac{L}{2} - y\right)^2}$$

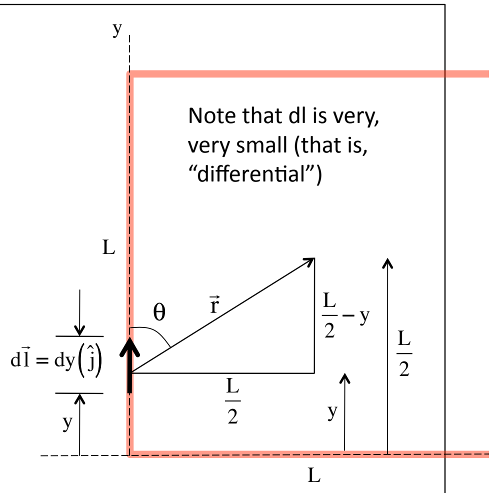
$$= \sqrt{\frac{L^2}{4} + \left(\frac{L^2}{4} - 2\frac{L}{2}y + y^2\right)}$$

$$= \left(y^2 - Ly + L^2\!/\!_2\right)^{1/2}$$

and

$$\sin\theta = \frac{L\!/\!_2 - y}{r}$$

$$= \frac{L\!/\!_2 - y}{\left(y^2 - Ly + L^2\!/\!_2\right)^{1/2}}$$



Note: We are going to integrate from 0 to L/2, then double that to get the net field due to one side. This seems obscure but circumvents the fact that as we get farther up the y-axis, the angle becomes greater than ninety degrees.

5.)

$$B = \int dB$$

$$= \frac{\mu_o i}{4\pi}\int \frac{d\vec{l} x \hat{r}}{r^2}$$

$$= 2\left[\frac{\mu_o i}{4\pi}\int_{y=0}^{L/2} \frac{dy\sin\theta}{\left[\left(y^2 - Ly + \frac{L^2}{2}\right)^{1/2}\right]^2}\right]$$

$$= \frac{2\mu_o i}{4\pi}\int_{y=0}^{L/2} \frac{dy\left[\dfrac{L\!/\!_2 - y}{\left(\frac{L^2}{2} + -Ly + y^2\right)^{1/2}}\right]}{\left(y^2 - Ly + \frac{L^2}{2}\right)}$$

$$= \frac{\mu_o i}{2\pi}\left[\int_{y=0}^{L/2} \frac{\left(L\!/\!_2\right)}{\left(y^2 - Ly + \frac{L^2}{2}\right)^{3/2}}dy - \int_{y=0}^{L/2} \frac{(y)}{\left(y^2 - Ly + \frac{L^2}{2}\right)^{3/2}}\right]$$

6.)

Slightly re-presented, we have:

$$B = \frac{\mu_o i}{2\pi}\left[\left(L\!/\!_2\right)\int_{y=0}^{L/2}\left(y^2 - Ly + \frac{L^2}{2}\right)^{-3/2} dy - \int_{y=0}^{L/2}(y)\left(y^2 - Ly + \frac{L^2}{2}\right)^{-3/2}\right]$$

If you are a wizard and love to do Calculus problems, you will undoubtedly find the solving of this a delightful amusement. I am not one of those people, so I’ve gone to my handy-dandy “Table of Integrals and Other Mathematical Data” (by Dwight) to find the solutions to the form of the integrals we have. They were:

$$\int\left(ay^2 + by + c\right)^{-3/2} dy = \frac{4ay + 2b}{\left(4ac - b^2\right)\left(ay^2 + by + c\right)^{1/2}}$$

and

$$\int y\left(ay^2 + by + c\right)^{-3/2} dy = \frac{\left(ay^2 + by + c\right)^{1/2}}{a} - \frac{b}{2a}\int\frac{1}{\left(ay^2 + by + c\right)^{1/2}}dy$$

Matching these expressions to our problem, we should end up with the same relationship that the book got by being clever with trig functions. If you have nothing else to do, try it (. . . not suggested for second semester seniors).

7.)